

Row Ideals and Fibers of Morphisms

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*Affectionately dedicated to Mel Hochster, who has been an inspiration to us for many years,
on the occasion of his 65th birthday.*

Abstract We study the fibers of projective morphisms and rational maps. We characterize the analytic spread of a homogeneous ideal through properties of its syzygy matrix. Powers of linearly presented ideals need not be linearly presented, but we identify a weaker linearity property that is preserved by taking powers.

1 Introduction

In this note we study the fibers of a rational map from an algebraic point of view. We begin by describing four ideals related to such a fiber.

Let $S = k[x_0, \dots, x_n]$ be a polynomial ring over an infinite field k with homogeneous maximal ideal \mathfrak{m} , $I \subset S$ an ideal generated by an $r+1$ -dimensional vector space W of forms of the same degree, and ϕ the associated rational map $\mathbf{P}^n \rightarrow \mathbf{P}^r = \mathbf{P}(W)$. We will use this notation throughout. Since we are interested in the rational map, we may remove common divisors of W , and thus assume that I has codimension at least 2.

A k -rational point q in the target $\mathbf{P}^r = \mathbf{P}(W)$ is by definition a codimension 1 subspace W_q of W . We write $I_q \subset S$ for the ideal generated by W_q . By a homogeneous presentation of I we will always mean a homogeneous free presentation of I with respect to a homogeneous minimal generating set. If $F \rightarrow G = S \otimes W$ is such a presentation, then the composition $F \rightarrow G \rightarrow S \otimes (W/W_q)$ is called the *generalized row* corresponding to q , and its image is called the *generalized row ideal* corresponding to q . It is the ideal generated by the entries of a row in the homogeneous presentation matrix after a change of basis. From this we see that the generalized row ideal corresponding to q is simply $I_q : I$.

The rational map ϕ is a morphism away from the algebraic set $V(I)$, and we may form the fiber (=preimage) of the morphism over a point $q \in \mathbf{P}^r$. The saturated ideal of the scheme-theoretic closure of this fiber is $I_q : I^\infty$, which we call the *morphism fiber ideal* associated to q .

The rational map ϕ gives rise to a *correspondence* $\Gamma \subset \mathbf{P}^n \times \mathbf{P}^r$, which is the closure of the graph of the morphism induced by ϕ . There are projections

$$\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{P}^r$$

and we define the *correspondence fiber* over q to be $\pi_1(\pi_2^{-1}(q))$. Since Γ is $\text{BiProj}(\mathcal{R})$, where \mathcal{R} is the Rees algebra $S[It] \subset S[t]$ of I , the correspondence fiber is defined by the ideal

$$(I_q t \mathcal{R} : (It)^\infty) \cap S = \bigcup_i (I_q I^{i-1} : I^i).$$

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This ideal describes the locus where I is not integral over I_q . It is not hard to see that our four ideals are contained, each in the next,

$$\begin{aligned} I_q &\subset I_q : I && \text{row ideal} \\ &\subset \bigcup_i (I_q I^{i-1} : I^i) && \text{correspondence fiber ideal} \\ &\subset I_q : I^\infty && \text{morphism fiber ideal .} \end{aligned}$$

In Section 2 we compare the row ideals, morphism fiber ideals, and correspondence fiber ideals.

In Section 3 we use generalized row ideals to give bounds on the analytic spread of I by interpreting the analytic spread as 1 plus the dimension of the image of ϕ .

Many interesting rational maps ϕ are associated as above to ideals I with linear presentation matrices—see for example Hulek, Katz and Schreyer [1992]. Thus we are interested in linearly presented ideals and their powers, which arise in the study of the graph. It is known that the powers of a linearly presented ideal need not be linearly presented. The first such examples were exhibited by Sturmfels [2000]; for a survey of what is known, see Eisenbud, Huneke and Ulrich [2006]. In Section 3 we also give criteria for birationality of the map, or for its restriction to a linear subspace of \mathbf{P}^n .

In Section 4 we generalize the notion of linear presentation (of an ideal or module) in various directions: A graded S -module M generated by finitely many elements of the same degree has *linear generalized row ideals* if the entries of *every* generalized row of a homogeneous presentation matrix for M generate a linear ideal, i.e., an ideal generated by linear forms. Obviously, any module with a linear presentation has this property, and we conjecture that the two notions are equivalent in the case of ideals. The corresponding conjecture is false for modules, but we prove it for modules of projective dimension one. The main result of the section implies the weak linearity property of powers mentioned in the abstract. It says, in particular, that if an ideal I has linear generalized row ideals, then every power of I has a homogeneous presentation all of whose (ordinary) rows generate linear ideals.

2 Comparing the notions of fiber ideals

Recall that the row ideal for a point q is always contained in the correspondence fiber ideal, which is contained in the morphism fiber ideal. If the row ideal is generated by linear forms (or, more generally, is prime) and does not contain I , then they are all equal. But in general the containments are both strict:

Example 2.1 Let $S = k[a, b, c, d]$, $J = (ab^2, ac^2, b^2c, bc^2)$, and $I = J + (bcd)$. One can check that I is linearly presented. Computation shows that the row ideal $J : I$ is (b, c) , while the correspondence fiber ideal is (a^2, b, c) and the morphism fiber ideal is the unit ideal $J : I^\infty = S$. We have no example of an \mathfrak{m} -primary ideal (regular morphism) where all three are different: in the examples we have tried, the correspondence fiber is equal to the morphism fiber. (Of course for any regular map all three are equal up to saturation, but we do not see why any two should be equal as ideals.)

Before stating the next result we recall that an ideal I in a Noetherian ring is said to be of *linear type* if the natural map from the symmetric algebra of I onto the Rees algebra

of I is an isomorphism. If I is of linear type, then I cannot be integral over any strictly smaller ideal, as can be seen by applying Theorem 4 on p.152 of Northcott and Rees [1954] to the localizations of I . We say that an ideal is *proper* if it is not the unit ideal.

Proposition 2.2 *If I has linear generalized row ideals, then every proper morphism fiber ideal is equal to the corresponding row ideal and hence generated by linear forms. If I is also of linear type on the punctured spectrum, then every proper correspondence fiber ideal is equal to the corresponding row ideal.*

Proof. Suppose that the morphism fiber ideal $I_q : I^\infty$ is not the unit ideal. In particular $I_q : I$ does not contain I . The required equality for the first statement is

$$I_q : I = I_q : I^\infty,$$

which follows because $I_q : I$ is linear, and thus prime.

Now suppose that I is of linear type on the punctured spectrum, and that the correspondence fiber ideal $H := \bigcup_i (I_q I^{i-1} : I^i)$ is proper. Set $K = I_q : I$, the row ideal. We must show $K = H$. Since $K \subset H$ we may harmlessly assume that K is not \mathfrak{m} , the homogeneous maximal ideal of S . By hypothesis the row ideal K is generated by linear forms, so it is prime. Since the localized ideals $(I_q)_K$ and I_K are not equal, and I_K is of linear type, it follows that I_K is not integral over $(I_q)_K$. Therefore H_K is a proper ideal. It follows that $H \subset K$, as required. ■

Example 2.3 The last statement of Proposition 2.2 would be false without the hypothesis that I is of linear type on the punctured spectrum. This is shown by Example 2.1.

Example 2.4 Let Q be a quadratic form in x_0, x_1, x_2 , and let F be a cubic form relatively prime to Q . The rational map defined by x_0Q, x_1Q, x_2Q, F has one morphism fiber (and correspondence fiber) ideal (Q) , though for a general point in the image both the morphism fiber ideal and the correspondence fiber ideal are linear. This example shows that in Theorem 4.1 of Simis [2004], the point p should be taken to be general.

3 How to compute the analytic spread and test birationality

The notions of row ideals and fiber ideals provide tests for the birationality of the map ϕ and lead to formulas for the analytic spread of the ideal I . In our setting, the *analytic spread* $\ell(I)$ of I can be defined as one plus the dimension of the image of the rational map ϕ . Its ideal theoretic significance is that it gives the smallest number of generators of a homogeneous ideal over which I is integral, or equivalently, the smallest number of generators of an ideal in $S_{\mathfrak{m}}$ over which $I_{\mathfrak{m}}$ is integral, see the corollary on p.151 of Northcott and Rees [1954].

Proposition 3.1

(a) *If q is a point in $\mathbf{P}^r = \mathbf{P}(W)$ such that $I_q : I^\infty \neq S$, then*

$$\ell(I) \geq 1 + \text{codim}(I_q : I^\infty);$$

(b) *If p is a general point in \mathbf{P}^n , then*

$$\ell(I) = 1 + \text{codim}(I_{\phi(p)} : I^\infty);$$

(c) *If there exists a point q so that the row ideal $I_q : I$ is linear of codimension n and does not contain I , then ϕ is birational onto its image. Moreover, ϕ is birational onto its image if and only if $I_{\phi(p)} : I^\infty$ is a linear ideal of codimension n for a general point p .*

Proof. Set $J = I_{\phi(p)}$. If the ideal $I_q : I^\infty$ is proper it cannot be \mathfrak{m} -primary, and hence defines a non-empty fiber of the morphism ϕ . On the other hand, $J : I^\infty$ is the defining ideal of a general fiber of the map. Thus the dimension formula and the semicontinuity of fiber dimension, Corollary 14.5 and Theorem 14.8(a) in Eisenbud [1995], show that

$$\text{codim}(I_q : I^\infty) \leq \text{codim}(J : I^\infty) = \dim \text{im}(\phi).$$

However, the latter dimension is $\ell(I) - 1$, proving parts (a) and (b).

The second assertion in (c) holds because the map is birational onto its image iff the general fiber is a reduced rational point.

We reduce the first assertion of (c) to the second one. Assume that the row ideal $I_q : I$ is linear of codimension n and does not contain I . Since $I_q : I$ is a prime ideal not containing I it follows that $I_q : I^\infty = I_q : I \neq S$. Thus the morphism fiber over q is not empty, and there exists a point $p \in \mathbf{P}^n$ with $q = \phi(p)$.

Now let T_0, \dots, T_r be variables over S and let A_1 denote the linear part of a homogeneous presentation matrix of I . We can write $(T_0, \dots, T_r) * A_1 = (x_0, \dots, x_n) * B$ for some matrix B whose entries are linear forms in the variables T_i with constant coefficients. The dimension of the space of linear forms in the row ideal corresponding to any point $\phi(p)$ is the rank of B when the coordinates of $\phi(p)$ are substituted for the T_i ; it is therefore semi-continuous in p . Thus for p general, the dimension of the space of linear forms in the ideal $I_{\phi(p)} : I$ is at least n , and then the same holds for $J : I^\infty$. As this ideal defines a nonempty fiber, it is indeed linear of codimension n . ■

Sometimes one can read off a lower bound on the analytic spread even from a partial matrix of syzygies. The following result is inspired by Proposition 1.2 of Hulek, Katz and Schreyer [1992].

Proposition 3.2 *With notation as above, suppose that A is a matrix of homogeneous forms, each of whose columns is a syzygy on the generators of I . Let A_q be the ideal generated by the elements of the generalized row of A corresponding to a point $q \in \mathbf{P}^r$. If there exists a prime ideal $P \in V(A_q)$ such that $A \otimes \kappa(P)$ has rank r , then $I_q : I^\infty \neq S$ and*

$$\ell(I) \geq 1 + \text{codim } A_q.$$

Proof. Since $A_q \subset I_q : I^\infty$, Proposition 3.1(a) shows that the second claim follows from the first one. To prove the first assertion, $I_q : I^\infty \neq S$, it suffices to verify that $(I_q : I^\infty)_P \neq S_P$.

As A_P contains an $r \times r$ invertible submatrix, and these relations express each generator of I_P in terms of the one corresponding to q , it follows that A_P is a full presentation matrix of the ideal I_P . Thus $(A_q)_P = (I_q : I)_P$. Furthermore, since I_P is generated by one element, and I has codimension at least 2 by our blanket assumption, it follows that $I_P = S_P$, whence $(A_q)_P = (I_q : I)_P = (I_q : I^\infty)_P$. On the other hand, $P \in V(A_q)$, so $(A_q)_P \neq S_P$, and we are done. ■

As in Proposition 1.2 of Hulek, Katz and Schreyer [1992], this gives criteria for birationality:

Corollary 3.3 *As in Proposition 3.2 suppose that $A \otimes \kappa(P)$ has rank r for some prime ideal $P \in V(A_q)$. The map ϕ is birational onto its image if A_q defines a reduced rational point in \mathbf{P}^n . The map ϕ , restricted to a general $\mathbf{P}^r \subset \mathbf{P}^n$ is birational (a Cremona transformation) if A_q defines a reduced linear space of codimension r in \mathbf{P}^n .*

Proof. Notice that $A_q \subset I_q : I \subset I_q : I^\infty$, where $I_q : I^\infty \neq S$ according to Proposition 3.2. Thus if A_q defines a reduced rational point in \mathbf{P}^n , then the row ideal $I_q : I$ is linear of codimension n and does not contain I . Thus ϕ is birational onto its image according to Proposition 3.1(c).

The second assertion follows from the first one, applied to the restriction of ϕ . \blacksquare

For other, related criteria for birationality we refer to Simis [2004].

4 Ideals with linear row ideals and their powers

We begin this section by clarifying the relation between these properties of an ideal or module: to have a linear presentation matrix, to have linear generalized row ideals, and to have *some* homogeneous presentation matrix all of whose row ideals are linear. Obviously, if a presentation matrix is linear then all its generalized row ideals are linear. However, the converse does not hold, at least for the presentation of modules with torsion. This can be seen by taking the matrix

$$\begin{pmatrix} s & t & t^2 \\ 0 & s & 0 \end{pmatrix}$$

for instance. However, we have:

Proposition 4.1 *If M is a graded S -module of projective dimension 1 generated by finitely many homogeneous elements of the same degree, and M has linear generalized row ideals, then M has a linear presentation.*

Proof. Reduce modulo n general linear forms, and use the Fundamental Theorem for modules over principal ideal domains. \blacksquare

Next, whenever an ideal has linear generalized row ideals, then obviously there is a presentation matrix with only linear row ideals. Again, the two concepts are not equivalent:

Example 4.2 We consider the ideal $I = (s^4, s^3t, st^3, t^4) \subset S = \mathbf{C}[s, t]$ corresponding to the morphism whose image is the smooth rational quartic curve in \mathbf{P}^3 . A homogeneous presentation of this ideal is given by

$$S^2(-5) \oplus S(-6) \xrightarrow{\begin{pmatrix} t & 0 & 0 \\ -s & 0 & t^2 \\ 0 & t & -s^2 \\ 0 & -s & 0 \end{pmatrix}} S^4(-4) \xrightarrow{(s^4 \ s^3t \ st^3 \ t^4)} S.$$

The row ideals of the second and third rows in this presentation are not linear. However, a change of basis in $S^4(-4)$, corresponding to a different choice of generators of I , makes them linear:

$$S^2(-5) \oplus S(-6) \xrightarrow{\begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ s-t & s-t & s^2-t^2 \\ -s+it & -is-t & s^2+t^2 \end{pmatrix}} S^4(-4) \xrightarrow{(F_0, \dots, F_3)} S,$$

where

$$\begin{aligned} F_0 &= -s(s-t)(s^2+t^2+(s+t)(s-it)) \\ F_1 &= -t(s-t)(s^2+t^2+(s+t)(is+t)) \\ F_2 &= st(s^2+t^2) \\ F_3 &= -st(s^2-t^2). \end{aligned}$$

Whereas powers of linearly presented ideals need not be linearly presented, the next result implies that having a homogeneous presentation with linear row ideals is a weak linearity property that is indeed preserved when taking powers.

Theorem 4.3 *If I has a homogeneous presentation matrix where at least one row ideal is linear of codimension at least $\ell(I) - 1$ and does not contain I , then each power of I has some homogeneous presentation matrix all of whose row ideals are linear of codimension $\ell(I) - 1$ and do not contain I .*

Proof. According to Proposition 3.1(b) for general $p \in \mathbf{P}^n$, the morphism fiber ideal $I_{\phi(p)} : I^\infty$ has codimension $\ell(I) - 1$, and hence the row ideal $I_{\phi(p)} : I$ has codimension at most $\ell(I) - 1$. Now one sees as in the proof of Proposition 3.1(c) that $I_{\phi(p)} : I$ is linear of codimension $\ell(I) - 1$ and does not contain I .

Let $E = V(I)$ be the exceptional locus of ϕ . For each $d \geq 1$ the rational map ϕ_d defined by the vector space of forms W^d is regular on $\mathbf{P}^n \setminus E$. For any point $p \in \mathbf{P}^n \setminus E$, the ideal of $\phi(p) \in \mathbf{P}(W)$ is generated by the vector space of linear forms $W_{\phi(p)}$, so the vector space of forms of degree d that it contains is $W_{\phi(p)}W^{d-1}$. Thus $(W^d)_{\phi_d(p)} = W_{\phi(p)}W^{d-1}$, and hence the row ideal corresponding to $\phi_d(p)$ is $I_{\phi(p)}I^{d-1} : I^d$.

We now show that for general p , the row ideal $I_{\phi(p)}I^{d-1} : I^d$ is linear of codimension $\ell(I) - 1$ and does not contain I . For trivial reasons we have

$$I_{\phi(p)} : I \subset I_{\phi(p)}I^{d-1} : I^d \subset I_{\phi(p)}I^{d-1} : I^\infty \subset I_{\phi(p)} : I^\infty.$$

By the above, $I_{\phi(p)} : I$ is a linear ideal of codimension $\ell(I) - 1$ and does not contain I . Hence

$$I_{\phi(p)} : I = I_{\phi(p)} : I^\infty,$$

and therefore

$$I_{\phi(p)} : I = I_{\phi(p)}I^{d-1} : I^d.$$

Let $\dim W^d = N + 1$. Because the image of ϕ_d is nondegenerate, $N + 1$ general points of \mathbf{P}^n correspond to the $N + 1$ rows of a presentation matrix of I^d , so we are done. ■

Corollary 4.4 *If I has linear presentation, or even just linear generalized row ideals, then every power of I has a homogeneous presentation matrix all of whose row ideals are linear of codimension $\ell(I) - 1$.*

Proof. According to Proposition 3.1(b), the homogeneous presentation matrix of I has a row ideal $I_q : I$ so that $\text{codim}(I_q : I^\infty) = \ell(I) - 1$. In particular $I_q : I^\infty \neq S$ and hence I is not contained in $I_q : I$. As $I_q : I$ is a linear ideal we conclude that $I_q : I = I_q : I^\infty$, which gives $\text{codim}(I_q : I) = \ell(I) - 1$. Now apply Theorem 4.3. ■

Proposition 4.5 *Every ideal has a homogeneous presentation where every row ideal has codimension at most $\ell(I) - 1$*

Proof. Take a homogeneous presentation whose rows correspond to the fibers through points of \mathbf{P}^n not in the exceptional locus. The row ideals are contained in the morphism fiber ideals, which have codimension at most $\ell(I) - 1$ according to Proposition 3.1(a). ■

5 Some open problems

We would very much like to know the answer to the following questions:

1. Can the homogeneous minimal presentation of an ideal I have linear generalized row ideals without actually being linear?
2. If ϕ is a regular map (that is, I is \mathbf{m} -primary), are the correspondence fiber ideals equal to the morphism fiber ideals? More generally, when are the correspondence fiber ideals saturated with respect to \mathbf{m} ?
3. If I is \mathbf{m} -primary and linearly presented, is every correspondence fiber ideal of the morphism defined by I^d either linear or \mathbf{m} -primary?
4. Find lower bounds for the number of linear relations I^d could have in terms of the number of linear relations on I . How close can one come to the known examples?

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